Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder

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SUMMARY

This paper is concerned with the problem of obtaining higher approximations to the flow past a sphere and a circular cylinder than those represented by the well-known solutions of Stokes and Oseen. Since the perturbation theory arising from the consideration of small non-zero Reynolds numbers is a singular one, the problem is largely that of devising suitable techniques for taking this singularity into account when expanding the solution for small Reynolds numbers.

The technique adopted is as follows. Separate, locally valid (in general), expansions of the stream function are developed for the regions close to, and far from, the obstacle. Reasons are presented for believing that these 'Stokes' and 'Oseen' expansions are, respectively, of the forms

$$\sum f_n(R)\psi_n(r,\theta) \quad \text{and} \quad \sum F_n(R)\Psi_n(Rr,\theta)$$

where (r,θ) are spherical or cylindrical polar coordinates made dimensionless with the radius of the obstacle, R is the Reynolds number, and f_{n+1}/f_n and F_{n+1}/F_n vanish with R. Substitution of these expansions in the Navier–Stokes equation then yields a set of differential equations for the coefficients ψ_n and Ψ_n, but only one set of physical boundary conditions is applicable to each expansion (the no-slip conditions for the Stokes expansion, and the uniform-stream condition for the Oseen expansion) so that unique solutions cannot be derived immediately. However, the fact that the two expansions are (in principle) both derived from the same exact solution leads to a 'matching' procedure which yields further boundary conditions for each expansion. It is thus possible to determine alternately successive terms in each expansion.

The leading terms of the expansions are shown to be closely related to the original solutions of Stokes and Oseen, and detailed results for some further terms are obtained.

1. INTRODUCTION

The problem of determining the steady flow past fixed bodies in a slow uniform stream of viscous incompressible fluid is an old one. It was first considered by Stokes (1851), and has been discussed subsequently by many authors. With very few exceptions, however, these authors have been almost entirely concerned with finding the flow past various shapes of body in the limit of zero Reynolds number. Yet many of the effects that arise when the Reynolds number is not negligibly small are also of considerable physical and mathematical interest. A number of phenomena in lubrication and the motion of small particles, for instance, depend critically on second-order effects arising from the inertia of the fluid. For practical purposes, a first approximation to these second-order effects would doubtless be quite adequate, but the purely mathematical difficulties encountered in their calculation raise fundamental questions concerning the general nature of expansions for flow fields at small Reynolds numbers. It is with such questions that the present paper is concerned.

The problem originally considered by Stokes (1851) was that of flow past a sphere, for which he obtained a solution by neglecting completely the inertia of the fluid. Later, Whitehead (1889) attempted to improve upon this solution by obtaining higher approximations to the flow when the Reynolds number is not negligibly small. The method proposed by Whitehead was the obvious one of using a lower-order approximation to calculate the inertia terms in the equation of motion, thus developing an iterative procedure. Since the boundary conditions at each stage of the iteration are independent of the Reynolds number, this procedure is clearly equivalent to assuming an expansion of the flow in powers of the Reynolds number. When this assumption is valid, and there are many slow motion problems for which it is*, there is little more to be said. But, as is now well known, the assumption is never valid in problems of uniform streaming. The particular difficulty encountered by Whitehead was that the second approximation to the velocity of flow past a sphere remains finite at infinity in a way which is incompatible with the uniform-stream condition. And higher approximations to the velocity distribution actually diverge at infinity. The assumption of an expansion in powers of the Reynolds number thus leads to a situation in which it is impossible to satisfy the boundary conditions of the problem in all terms except the leading one. This mathematical phenomenon appears to be common to all problems of uniform streaming past bodies of finite length-scale, and is sometimes referred to as 'Whitehead's paradox'.

The paradox has, of course, long since been resolved. Both its physical origin and a mathematical device for overcoming the associated difficulties were pointed out by Oseen (1910). Since any body moving steadily through a viscous fluid must experience some resistance, consideration of the momentum flux across a large surface surrounding the body shows that the magnitude of the disturbance to the uniform velocity of the stream

* Apart from some finer qualifications, the necessary and sufficient condition for the validity of this assumption in problems of steady flow is that the velocity should fall to zero at a great distance from the boundaries generating the flow not less rapidly than the reciprocal of the distance. The assumption is thus valid for the flow generated by a slowly rotating sphere.

cannot everywhere fall to zero more rapidly than the inverse square of the distance from the body. But, for so large a disturbance as this, the acceleration of the fluid at a great distance cannot be negligible by comparison with the local viscous force. For the former, being almost entirely due to the convective effect of the stream, is a constant multiple of the first derivative of the velocity, whereas the latter is a multiple of the second derivative of the velocity. Thus the viscous force can be dominant only if the decay of the disturbance is exponentially rapid. Stokes's theory is therefore not self-consistent at a great distance from the body, and it is not surprising that the procedure adopted by Whitehead should lead to further inconsistencies. In more mathematical terms, the perturbation represented by a small non-zero Reynolds number has a singularity at infinity (in space), and Stokes's solution does not provide a uniformly valid approximation to all the required properties of the flow.

It is a straightforward matter to show that Stokes's solution does not break down until the region in which the flow is nearly a uniform stream has been reached, so that the solution does provide a uniformly valid approximation to the total velocity distribution (and consequently a valid approximation to many bulk properties of the flow, such as the resistance). It is only the derivatives of the velocity at a great distance that are seriously in error. But, of course, this error is crucial in the problem of obtaining a second approximation to the flow, since the neglected inertia terms in the equations of motion involve velocity gradients. Nor can Whitehead's procedure be used to derive a locally valid second approximation to the flow in the region not far from the sphere. It is true that this procedure produces a correct differential equation for the second approximation, but the spatial restrictions placed upon its validity prohibit the use of the outer boundary conditions, so that a unique solution cannot be derived. In fact, as in all singular perturbation problems, a uniformly valid approximation to the neglected terms in the governing equation is a necessary prerequisite for the determination of a second approximation to the solution *anywhere* in the field.

Fortunately, as was shown by Oseen (1910), the determination of a uniformly valid first approximation to the velocity and all its derivatives is itself a linear problem which may be solved analytically. The circumstance, already mentioned, that the inertia terms are important only in the region where uniform-stream conditions have been almost attained permits a linear approximation to be made which yields the well-known Oseen equation. The very interesting description of the asymptotic flow field given by the solutions of this equation now occupies an important place in the theory of viscous motion. In the present paper, however, we are more interested in the use of these solutions as tools in the problem of finding higher approximations.

We may also note that the relevant solution of Oseen's equation provides a uniformly valid approximation to the velocity and all its derivatives in the two-dimensional flow past an infinite cylinder of finite cross-sectional length-scale, though Stokes's approximation yields no solution in this case. The first such solution to be obtained was that for flow past a circular cylinder (Lamb 1911), and a more quantitative summary of these results and those for flow past a sphere is given in § 2.

In both the two- and three-dimensional cases, therefore, the solutions of Oseen's equation provide an adequate starting point for the determination of higher approximations to the flow. However, apart from the recent paper by Lagerstrom & Cole (1955), no work on this problem seems to have been published. It is true that a great deal of effort has been expended in finding higher approximations on the assumption that Oseen's equation, rather than the Navier–Stokes equation, is the exact governing equation; but this problem is not strictly relevant. Oseen's equation does contain the Reynolds number as a free parameter (an obviously necessary consequence of the uniform validity of Oseen's approximation), but the idea of solving the equation to a higher order of approximation in the small Reynolds number than that involved in its derivation is of limited value. The justification usually given is that Oseen's equation and the Navier–Stokes equation are qualitatively similar, so that solutions of the former might be expected to yield qualitative information about solutions of the latter for all Reynolds numbers. And on these grounds, a few basic solutions, such as Goldstein's (1929) solution for the sphere, are surely worth while. But the problem as a whole has probably received far more attention than it deserves.

In principle, the problem of obtaining higher approximations to the real flow is not appreciably more difficult than that mentioned above. For there seems little reason to doubt that Whitehead's iterative method, using Oseen's equation rather than Stokes's equation would yield an expansion, each successive term of which would represent a uniformly valid higher approximation to the flow. In each step of the iteration a lower-order approximation would be used to calculate those particular inertia terms that are neglected in Oseen's equation and the resulting inhomogeneous form of Oseen's equation would be solved, to the relevant degree of accuracy, for the boundary conditions appropriate to the problem. The expansion generated in this way would seem to be the most economic expansion possible, in the sense that the partial sums of any order contain all the legitimate, and no redundant, information about the whole flow field.

Nevertheless, it is the primary object of the present paper to describe in detail an alternative procedure which in many ways is more satisfactory. This alternative procedure involves simultaneous consideration of locally valid (in general) expansions close to, and far from, the singularity of the perturbation. These expansions may be called 'Oseen' and 'Stokes' expansions, respectively, since their leading terms are closely related to the original approximations of these authors. The Stokes expansion is a straightforward expansion of the kind envizaged by Whitehead. It is an expansion in the Reynolds number for fixed values of the space coordinates (made dimensionless with the finite length-scale of the body). For the Oseen expansion, on the other hand, the coordinate system is first strained, by a factor depending upon the Reynolds number, in such a way that the length-scale of variations in the asymptotic flow pattern at a great distance from the body is finite in terms of the new coordinates. In this new coordinate system, the length-scale of the body is very small, and the singularity of the perturbation is removed to the origin of coordinates (inside the body). The Oseen expansion is then an expansion in the Reynolds number of fixed values of the new coordinates. The connection between this expansion and Oseen's work is evident when one remembers that both the inertia and viscous forces at a great distance depend *linearly* on the disturbance to the stream. For the fact that these forces are of a comparable order of magnitude is then made to appear 'natural' by the choice of length-scale referred to above.

On grounds of expediency alone, the use of Stokes and Oseen expansions is preferable to the use of an expansion which is generated by uniformly valid successive approximations. In the first place, their mathematical structure is a great deal simpler. In fact, they are usually power series, or simple extensions of power series, in the Reynolds number; whereas a uniformly valid approximation necessarily depends upon the Reynolds number in a complicated manner since it involves functions of both the 'strained' and 'unstrained' coordinate systems. Moreover, uniformly valid approximations *per se* are not usually of much physical interest (apart, perhaps, from a first approximation, which gives an overall description of the singularity of the perturbation). In the present problem, for instance, it is the Stokes expansion that gives virtually all the physically interesting information. Questions of uniformity arise only in connection with the proper derivation of this expansion, and there is therefore some point in attempting to cast such questions—and techniques for answering them—in terms of the expansion itself.

However, the attitude adopted by Lagerstrom & Cole (1955), who also discuss in general terms the use of Stokes and Oseen expansions in the problem of the present paper, is rather more fundamental. They point out that, when dealing with asymptotic expansions for small Reynolds numbers, it is wise to restrict attention to those expansions that can (in principle) be derived from the exact solution by the application of formal limit processes which may be defined *à priori*. For it is then a relatively straightforward matter to discuss such questions as the error involved in any particular partial sum, or the domain of uniform validity of the expansion. The Stokes and Oseen expansions are of this type since they may be derived by the limiting processes described above. Thus, if R is the Reynolds number and x is the (dimensionless) position vector, the limiting process $R \to 0$ for either fixed x or fixed Rx defines, respectively, the Stokes or Oseen limit. The application of these limits to the difference between the exact solution and the nth partial sum then yields the $(n+1)$th term of the relevant expansion.

An expansion in terms of uniformly valid successive approximations, on the other hand, cannot normally be derived from the exact solution in this way. It can be derived from the exact solution by inspection, when the detailed form of the solution is known; or it can be defined, as has been done above, in terms of the iterative process that is to be applied to the governing equation. However, both these concepts are very cumbersome, and since uniformly valid approximations may always be constructed from a simultaneous knowledge of the Stokes and Oseen expansions, it is more satisfactory to proceed in this latter way rather than *vice versa.*

The problem thus reduces to that of determining the proper boundary conditions for the individual terms of the Stokes and Oseen expansions. At the regular end of the range of validity of these expansions (the body for the Stokes expansion, and infinity for the Oseen expansion), the boundary conditions are the physically obvious ones. At the singular end, however, the physical boundary conditions are irrelevant and it is necessary to use the fact that the Stokes and Oseen expansions are different forms of the same function. This leads to a matching of the expansions which is of such a kind that it becomes possible to derive alternately successive terms in each expansion. These matching conditions have already been described in general terms by Lagerstrom & Cole (1955) and a detailed account of the procedure is now given in § 3 and § 4.

For the sake of simplicity, the paper deals only with flow past a sphere and a circular cylinder (treated respectively in § 3 and § 4), since these special cases appear to illustrate most* of the pertinent ideas.

2. The approximations of Stokes and Oseen

2.1. *Flow past a sphere*

In Stokes's original treatment (1851) of slow streaming past a sphere, the inertia of the fluid is neglected completely, so that the Navier–Stokes equation reduces to

$$0 = -\operatorname{grad} p + \nu \nabla^2 \mathbf{u}, \tag{2.1}$$

where p is the kinematic pressure, ν is the kinematic viscosity, and $\mathbf{u}$ is the velocity vector. It is then a straightforward matter to obtain, in terms of a Stokes stream function, the integral of this equation that satisfies the no-slip condition at the sphere and the uniform-stream condition at infinity. If U is the velocity of the undisturbed stream, and a is the radius of the sphere, this integral for the Stokes stream function $Ua^2\psi$ is

$$\psi = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta, \tag{2.2}$$

where the origin of polar coordinates (ar, θ, ϕ) is taken at the centre of the sphere, the line $\theta = 0$ being in the direction of the undisturbed stream.

* Though certainly not all. For a recent account of the important effects of asymmetry in two-dimensional flow past a cylinder, see Imai (1951). The corresponding three-dimensional problem appears not to have been considered.

The individual terms in (2.2) are, successively, a uniform stream, a so-called 'stokeslet', and a doublet. It is, of course, only the stokeslet which contributes to the vorticity of the flow.

The values of r at which Stokes's approximation breaks down may now be found by using (2.2) to calculate the neglected terms. When r is large, the dominant inertia terms are the convective accelerations arising from the interaction between the uniform stream and the stokeslet, and are of order U^2/ar^2. The viscous term, on the other hand, is of order $\nu U/a^2r^3$ in this part of the flow. The Stokesian theory therefore breaks down when

$$Rr = O(1), \tag{2.3}$$

where R is the small Reynolds number Ua/ν. It is important to note in this criticism that the dominant inertia terms arise from the *rotational* velocity field due to the stokeslet, and cannot be represented as the gradient of a scalar. If the latter had been possible, the error arising from these terms could have been absorbed into the solution for the pressure field without affecting the solution (2.2) which is the *general* solution for a conservative distribution of viscous forces.

According to (2.3), the solution (2.2) approaches arbitrarily close (for sufficiently small R) to the conditions of a uniform stream before the theory breaks down. Stokes's solution is therefore actually a uniform approximation to the total velocity distribution*. However, it is clearly not a uniform approximation to the disturbance of the uniform stream, or, equivalently, to the distribution of velocity gradients. As noted in § 1, it is for this reason that the solution cannot be used to obtain a second approximation in the manner attempted by Whitehead (1889).

The idea behind Oseen's (1910) technique for obtaining a uniform approximation to the disturbance of the stream is to take inertia forces into account in the region where they are comparable with viscous forces, but neglect them in the Stokesian region of the flow. Thus, since the flow is nearly a uniform stream in the former region, the appropriate equation is

$$\mathbf{U}\,.\,\mathrm{grad}\,\mathbf{u} = -\mathrm{grad}\,p + \nu\nabla^2\mathbf{u}, \tag{2.4}$$

where the vector $\mathbf{U}$ represents the uniform stream. The left-hand side of (2.4) is, of course, negligible throughout the region in which Stokes's approximation is valid. It may be noted in passing that the equation (2.4) is formally the same as the equation which would be obtained if the velocity distribution were written in the form $\mathbf{U}+\mathbf{u}$ and the Navier–Stokes equation were linearized in the disturbance velocity $\mathbf{u}$. However, this interpretation is conceptually wrong and can lead to erroneous or misleading conclusions such as Lamb's statement (1932, p. 610) that Oseen's theory is less accurate

* In this paper, we consider $\mathbf{u}^*$ to be a uniform approximation to $\mathbf{u}$ if, as $R \to 0$, $\mathbf{u}-\mathbf{u}^* = o(\mathbf{u})$ for all values of $\mathbf{x}$. This condition may clearly be violated in a trivial sense near the regular zeros of $\mathbf{u}$. However, in any particular problem, we are interested in the condition only near the singularities of the perturbation.

than Stokes's in the neighbourhood of the sphere (where the boundary condition $\mathbf{u} = -\mathbf{U}$ would make nonsense of such a linearization). The left-hand side of (2.4) is not intended to be a uniform approximation to the inertia terms, and the difference between Oseen's and Stokes's theory in the neighbourhood of the sphere is of a small order which neither theory is entitled to discuss.

The derivation of the appropriate exact solution of Oseen's equation (2.4) is a matter of some difficulty (see Goldstein 1929). Fortunately, however, there is no justification (in the present investigation) for finding a solution which satisfies the boundary conditions to a higher order of approximation than that involved in the governing equation, and it is a relatively simple matter to show that the solution given by Oseen himself is an adequate approximation. In terms of the dimensionless stream function ψ, this solution is

$$\psi = \frac{1}{4}\left(2r^2 + \frac{1}{r}\right)\sin^2\theta - \frac{3}{2R}(1+\cos\theta)(1-e^{-\frac{1}{2}Rr(1-\cos\theta)}). \tag{2.5}$$

The function (2.5) certainly satisfies the differential equation (2.4) and the relevant boundary condition at infinity. Moreover, when r is of order unity, the function becomes

$$\psi = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta + O(R),$$

which agrees with Stokes's solution, and consequently satisfies the boundary condition on the sphere, to an adequate approximation.

That (2.5) provides a uniform approximation to the disturbance of the stream follows immediately from the manner in which it has been obtained, and it is interesting to observe the nature of the non-uniformity associated with Stokes's solution. According to (2.5), the symmetric flow due to the stokeslet changes, at large values of Rr, into a simple source described by the stream function

$$\psi \sim -\frac{3}{2R}(1+\cos\theta),$$

whose mass flux is supplied asymmetrically by an inflow along a narrow wake defined, in an order-of-magnitude sense, by

$$Rr(1-\cos\theta) = O(1).$$

The function that describes this transition is a function of Rr and θ, and this strained coordinate system will appear as the natural one for the entire Oseen expansion (see § 3.2).

Finally, it should be noted that Oseen's equation (2.4) is only a first approximation to the governing equation and cannot be used as such to derive second approximations to any property of the flow. Thus Oseen's derivation (1913) of a second approximation to the drag coefficient of a sphere, namely

$$C_D = \frac{D}{U^2a^2} = \frac{6\pi}{R}(1 + \tfrac{3}{8}R), \tag{2.6}$$

where D is the kinematic drag, would seem to be invalid, though the result is in fact correct (see § 3.4). Strictly, Oseen's method gives only the leading term of (2.6) and is scarcely to be counted as superior to Stokes's method for the purpose of obtaining the drag.

2.2. *Flow past a circular cylinder*

The two-dimensional case exhibits profound differences. For slow uniform streaming motion past a circular cylinder, there is no solution to Stokes's equation (2.1) that remains finite as r becomes indefinitely large and that satisfies the no-slip condition on the cylinder. Hence, unlike the three-dimensional case, there is no solution to Stokes's equation that provides a uniform approximation to the total velocity distribution; it is therefore not immediately clear that Oseen's equation (2.4) will provide a uniform approximation to even the total velocity distribution.

Let ar and θ refer in this section to the two-dimensional radial and angular polar coordinates respectively, the line $\theta = 0$ being the positive direction of the stream, and let $Ua\psi$ be the Lagrange stream function, where a is the radius of the cylinder and U the uniform streaming velocity. If, now, we seek that solution of Stokes's equation which satisfies the no-slip condition on the cylinder $r = 1$, which contains a term $\sin\theta$ (in view of the uniform stream condition at infinity), and which diverges least rapidly as r becomes indefinitely large, we obtain

$$\psi = C\left(\frac{1}{r} - r + 2r\log r\right)\sin\theta. \qquad (2.7)$$

For very large r, the solution (2.7) is dominated by the rotational term $r\log r$, and, at first sight, appears wholly unsatisfactory because of this logarithmic divergence. However, by substituting (2.7) into the full equation of motion, we find that the inertial forces neglected are of order $C^2(\log r)/r^2$, and the viscous forces of order C/Rr^3. These will be comparable when

$$CRr\log r = O(1), \qquad (2.8)$$

and we should not expect the Stokes solution (2.7) to be valid beyond a value of r given by (2.8). In other words, while (2.7) may be an adequate representation of the flow relatively close to the cylinder, it cannot represent a uniform approximation to the total velocity distribution.

It is possible, however, to write the approximation (2.7) in such a way that this non-uniformity appears less severe than might at first sight be supposed. Thus we may write (2.7) in the form

$$\psi = C\left(-2r\log\{f(R)\} + 2r\log\{rf(R)\} - r + \frac{1}{r}\right)\sin\theta, \qquad (2.9)$$

when $f(R)$ is an arbitrary function of R. For $f(R) \ll 1$, and $rf(R)$ of order unity, the dominant term in (2.9) will be $-2C\log\{f(R)\}r\sin\theta$. If this is to represent the external flow, namely a uniform stream $r\sin\theta$, then we can put $C = -1/2\log\{f(R)\}$. Further, by substituting this value of C and $r = 1/f(R)$ into (2.8) we get

$$R/f(R) = O(1). \tag{2.10}$$

Thus for $f(R) = R$, the Stokes solution (2.9) leads to a uniform stream of order unity in that very region where Stokes's equation ceases to be valid. It is this apparently fortuitous behaviour that suggests that the external uniform stream condition has been reached before the Stokes approximation breaks down. The logarithmic term in (2.7) apparently plays an important role in making the solution (2.7) as nearly uniform as the approximation allows, and it is interesting to note that a uniform approximation is obtained by making only a slight change in the solution (2.7). Thus the form

$$\psi = -\frac{1}{2\log R}\left(\frac{1}{r} - r + 2r\log\frac{r}{1+Rr}\right)\sin\theta \tag{2.11}$$

is such that it tends to $r\sin\theta$ as r tends to infinity, and also only differs from the Stokes solution (2.7) by terms of order R when r is of order unity. The form (2.11) may be regarded as an analogue of (2.2) in the three-dimensional case, and for similar reasons, we are led to expect that Oseen's equation (2.4) will provide a uniform approximation to the disturbance stream function.

In terms of the stream function, Oseen's equation becomes

$$\left(\nabla_r^2 - R\frac{\partial}{\partial x}\right)\nabla_r^2\,\psi = 0 \tag{2.12}$$

where $x = r\cos\theta$. Faxén (1927) and later Tomotika & Aoi (1950) have solved this equation exactly to satisfy the no-slip condition on the cylinder and the uniform-stream condition at infinity. However, as has been explained for the case of the sphere, there is no point in solving the linear equation (2.12) to a greater degree of approximation than that of the inertial terms neglected by substituting the Oseen equations for the Navier–Stokes equations, and so the simpler solution given by Lamb (1911) is as good an approximation as it is possible to obtain from the linearized equation. In fact, by writing Lamb's solution in terms of ψ, in the form

$$\psi = \left(r + \frac{1}{2B_0 r}\right)\sin\theta - \sum_{n=1}^{\infty}\frac{1}{2B_0}\phi_n(\tfrac{1}{2}Rr)\frac{r\sin n\theta}{n} + O(B_0^{-2}), \tag{2.13}$$

where
$$B_0 = \tfrac{1}{2} - \gamma - \log\tfrac{1}{4}R \tag{2.14}$$
and
$$\phi_n = 4K_1 I_n + 2K_0(I_{n+1} + I_{n-1}), \tag{2.15}$$

the K_p and I_n being modified Bessel functions, we get a uniform approximation to the disturbance stream function over the entire flow. Within the Stokes region (2.13) becomes, to the approximation involved, a Stokes solution, while far from the cylinder it represents a characteristic wake flow similar to that described for the sphere. But whereas the solution for the sphere is correct to terms of order R, the solution for the cylinder is correct only to terms of order $(\log R)^{-1}$.

3. General expansions for flow past a sphere

In this section we consider the problem of how to obtain higher approximations to the velocity distribution in flow past a sphere. The disadvantages of applying a uniformly valid iterative process to Oseen's equation (2.4) have already been noted in § 1. The terms of the resulting expansion, of which the leading term (2.5) is typical, involve the Reynolds number in far too heterogeneous a manner to make the significance of the expansion at all clear. Instead, therefore, we employ the more homogeneous Stokes and Oseen expansions are described below.

3.1. *The Stokes and Oseen expansions*

In the Stokes region of the flow, that is where $r = O(1)$, we write the governing equation for the stream function ψ in its customary form, namely

$$\frac{1}{r^2}\frac{\partial(\psi, D_r^2\psi)}{\partial(r,\mu)} + \frac{2}{r^2}D_r^2\psi L_r\psi = \frac{1}{R}D_r^4\psi, \tag{3.1}$$

where

$$\mu = \cos\theta, \tag{3.2}$$

$$D_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}, \tag{3.3}$$

$$L_r = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \mu}, \tag{3.4}$$

and assume an expansion of the form

$$\psi = f_0(R)\psi_0(r,\mu) + f_1(R)\psi_1(r,\mu) + \ldots, \tag{3.5}$$

where

$$f_{n+1}(R)/f_n(R) \to 0 \quad \text{as} \quad R \to 0. \tag{3.6}$$

The expansion (3.5) should be regarded as the expansion of the exact solution $\psi(r, \mu, R)$ for small values of R at a *fixed* value of r. The assumption that the expansion takes the form indicated is thus equivalent to the very mild assumption that there is no singular dependence on R in the finite part of the field (such as arises, for instance, in connection with the formation of shear layers at *large* values of R). We refer to (3.5) as the Stokes expansion.

Since we require the magnitude of the velocity to be everywhere bounded, we may write

$$f_0(R) \equiv 1 \tag{3.7}$$

without loss of generality. Formal allowance is thus made for the possibility that $\psi_0 \equiv 0$, though in fact this is not necessary, since it is evident that ψ_0 must be Stokes's solution.

The expansion (3.5) is required to satisfy the differential equation (3.1) and the no-slip condition on the sphere. Since the expansion is invalid at large values of r, the uniform-stream condition at infinity must be replaced by the requirement that the expansion should be perfectly matched to an expansion which *is* valid in the outer region.

The reason for the breakdown of the Stokesian theory is that inertia and viscous forces become comparable at large values of r. This suggests that, for the outer expansion, we should find a transformation which removes the Reynolds number from the governing equation, thereby suiting the coordinate system to the fact that all terms in the equation are of a comparable order of magnitude. There are, of course, many such transformations, but for the purpose in hand we assume that they are all essentially equivalent and select the simplest, which is an isotropic straining of the coordinate system and a simple scaling of the stream function. Thus we introduce the variables

$$\rho = f(R)r, \qquad \Psi = g(R)\psi, \tag{3.8}$$

and, when the governing equation is expressed in terms of these variables, the condition that the R should not appear is

$$Rf(R) = g(R). \tag{3.9}$$

Further, since the (dimensionless) velocity must be of order unity in the region of validity of the expansion now sought, that is, where ρ and Ψ are of order unity, we must have

$$f^2(R) = g(R). \tag{3.10}$$

In this way we obtain the Oseen variables

$$\rho = Rr, \qquad \Psi = R^2\psi, \tag{3.11}$$

in terms of which the governing equation (3.1) becomes

$$\frac{1}{\rho^2}\frac{\partial(\Psi, D_\rho^2\Psi)}{\partial(\rho,\mu)} + \frac{2}{\rho^2}D_\rho^2\Psi L_\rho\Psi = D_\rho^4\Psi, \tag{3.12}$$

where D_ρ^2 and L_ρ are the same operators as (3.3) and (3.4), but with r replaced by ρ.

The expansion in the outer region, which we call the Oseen expansion, is now assumed to take the form

$$\Psi = \Psi_0(\rho,\mu) + F_1(R)\Psi_1(\rho,\mu) + F_2(R)\Psi_2(\rho,\mu) + \dots, \tag{3.13}$$

where
$$F_{n+1}(R)/F_n(R) \to 0 \quad \text{as} \quad R \to 0. \tag{3.14}$$

That the leading term should be independent of R is, of course, implicit in the choice of the Oseen variables (3.11). The structure of the remainder of the expansion then rests upon assumptions similar to those made for the Stokes expansion. In the present case, however, the absence of a singular dependence on ρ and μ (except at $\rho = 0$) as $R \to 0$ is not intuitively obvious, and the plausibility of the expansion (3.13) depends on the demonstration that it is possible to satisfy the governing equations and boundary conditions with such an expansion.

The expansion (3.13) is required to satisfy the differential equation (3.12) and the uniform-stream condition at infinity. On the other hand, the no-slip condition on the sphere, which in the new coordinate system has shrunk to a very small sphere of radius R (thus introducing the Reynolds number into the boundary conditions rather than the differential equation), is replaced by the condition that (3.13) should be matched to the Stokes expansion (3.5) at small values of ρ.

The matching conditions follow from the fact that the Stokes and Oseen expansions must be related in some way, since they are both expansions of the same function for small values of R, even though one expansion does not in general determine the other uniquely. The nature of this relation, and hence the conditions themselves, may be found from the following rather intuitive argument. The common features of the Stokes expansions of the infinitely many functions that all have the same Oseen expansion ought to be possessed by the 'Stokes expansion' of the Oseen expansion itself. Thus, if the Oseen expansion

$$\sum F_n(R)\Psi_n(Rr,\mu)$$

is formally expanded about $R = 0$ for fixed values of r (by expanding the $\Psi_n(\rho,\mu)$ for small values of ρ, and re-arranging the terms), the resulting expansion,

$$\sum g_n(R)\phi_n(r,\mu),$$

must be closely related to the Stokes expansion (3.5) of the actual flow. In fact, the only reason why these last two expansions might not be identical is that the Stokes coefficients $\psi_n(r,\mu)$ might involve terms (like $e^{-r} = e^{-\rho/R}$) which are important in the Stokes region but which are so small when r is large that they do not contribute to any term of the Oseen expansion. Thus, taking all such possibilities into account, it seems that we must have $f_n(R) = g_n(R)$ and that $\psi_n(r,\mu)$ and $\phi_n(r,\mu)$ must have the same asymptotic expansion for large values of r. The expansions of the Oseen coefficients for small values of ρ thus determine uniquely the expansions of the Stokes coefficients for large values of r (and *vice versa*).

In practice, of course, it is necessary to solve for one term at a time, and the procedure for, say, a Stokes term is then as follows. The general solution for this term that satisfies the no-slip condition on the sphere is first obtained from the relevant differential equation and is then expanded as an Oseen expansion. The individual terms in this expansion are then compared with the corresponding terms that have previously been calculated in the Oseen expansion of the full solution. According to the conditions derived above, the former terms must occur explicitly in the expansions of the latter terms for small values of ρ. A comparison of coefficients then uniquely determines the arbitrary constants in the general solution for the Stokes term. The analogous procedure for the calculation of an Oseen term is obvious.

3.2. *The leading terms of the expansions*

It is clear from the conventional approach described in §2.1 that the leading terms of the expansions (3.5) and (3.13) are, respectively, Stokes's solution

$$\psi_0 = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)(1-\mu^2), \tag{3.15}$$

and a uniform stream

$$\Psi_0 = \tfrac{1}{2}\rho^2(1-\mu^2). \tag{3.16}$$

These expressions both satisfy their respective differential equations and boundary conditions. Moreover, the matching of these leading terms is particularly trivial. When (3.15) is written in terms of the Oseen variables (3.11), its contribution to Ψ is seen to be

$$\frac{1}{4}\left(2\rho^2 - 3R\rho + \frac{R^3}{\rho}\right)(1-\mu^2), \tag{3.17}$$

so that only the uniform-stream term contributes to Ψ_0. The boundary conditions for Ψ_0 are therefore the same at $\rho = 0$ and $\rho = \infty$, and the exact solution of the non-linear equation for Ψ_0 is the uniform stream (3.16). This result is, of course, merely a re-statement of the fact that Stokes's solution gives a uniform approximation to the total velocity distribution.

Although the forms of the leading terms are already known, it is instructive to consider their derivation *ab initio*, if for no other reason than to prepare the ground for the rather more difficult two-dimensional case. In this connection it should be noted that it is sufficient to determine either of the solutions (3.15) and (3.16), since adequate boundary conditions are then available for the determination of the other.

Now, there appears to be a simple physical argument for the solution (3.16). The Oseen variable ρ is, by definition,

$$Rr = \frac{Ua}{\nu}\frac{r'}{a} = \frac{Ur'}{\nu}, \tag{3.18}$$

where r' is the (dimensional) radial distance from the centre of the sphere; and this is independent of a. For fixed values of U and ν, therefore, a fixed value of ρ corresponds to a fixed position in space. Hence, if the limiting process $R \to 0$ is interpreted as the limiting process $a \to 0$, it follows that the flow at the fixed point under consideration must ultimately be that of the undisturbed stream. It is true that the argument makes some assumptions about the magnitude of the disturbance caused by a body of zero size, implying, for instance, that the total force on the sphere vanishes with its radius, but such assumptions can be accepted with some assurance.

The derivation of the leading term of the Stokes expansion, ψ_0, is then straightforward. The equation for ψ_0 is

$$D_r^4\psi_0 = 0, \tag{3.19}$$

and it is not difficult to show that the general regular solution of this equation, that vanishes at $\mu = \pm 1$ has a double zero at $r = 1$, is

$$\psi_0 = \sum_{n=1}^{\infty} [A_n\{(2n-1)r^{n+3} - (2n+1)r^{n+1} + 2r^{-n+2}\} + \\ + B_n\{2r^{n+1} - (2n+1)r^{-n+2} + (2n-1)r^{-n}\}]Q_n(\mu), \tag{3.20}$$

where A_n and B_n are constants, and

$$Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu)\,d\mu, \tag{3.21}$$

in which $P_n(\mu)$ is the Legendre polynomial of degree n. When (3.20) is expressed in terms of the Oseen variable ρ, its contribution to Ψ is found to be

$$\sum_{n=1}^{\infty} [A_n\{(2n-1)R^{-n-1}\rho^{n+3}-(2n+1)R^{-n+1}\rho^{n+1}+2R^n\rho^{-n+2}\}+ \\ +B_n\{2R^{-n+1}\rho^{n+1}-(2n+1)R^n\rho^{-n+2}+(2n-1)R^{n+2}\rho^{-n}\}]Q_n(\mu), \quad (3.22)$$

and the requirement that this contribution should not contain terms of greater order than unity then gives

$$\left.\begin{aligned} A_n &= 0, \\ B_n &= 0 \qquad (n \geqslant 2), \end{aligned}\right\} \quad (3.23)$$

Further, since

$$Q_1(\mu) = -\tfrac{1}{2}(1-\mu^2), \quad (3.24)$$

the requirement that the term of order unity should represent the uniform stream (3.16) gives

$$B_1 = -\tfrac{1}{2}. \quad (3.25)$$

Thus, Stokes's solution (3.15) is recovered.

3.3. *The second term of the Oseen expansion*

Since the leading term of the Oseen expansion is a uniform stream, the equation for the second term Ψ_1 must, in effect, be Oseen's equation (2.4). In terms of the stream function, this equation is

$$\frac{1-\mu^2}{\rho}\frac{\partial D_\rho^2\Psi_1}{\partial\mu}+\mu\frac{\partial D_\rho^2\Psi_1}{\partial\rho} = D_\rho^4\Psi_1. \quad (3.26)$$

It may be noted that this equation now appears naturally as the equation for the perturbation of a uniform stream, which appears to be at variance with the view taken in § 2.1. The reason is, of course, that we are no longer dealing with a technique for obtaining a uniform approximation to the solution.

Equation (3.26) may be solved by the method used by Goldstein (1929). The transformation

$$D_\rho^2\Psi_1 = e^{\frac{1}{2}\rho\mu}\Phi \quad (3.27)$$

reduces the equation to

$$(D_\rho^2-\tfrac{1}{4})\Phi = 0, \quad (3.28)$$

and the general solution that vanishes at infinity and $\mu = \pm 1$ (the latter condition follows immediately from (3.3) of Ψ vanishes at $\mu = \pm 1$) is easily found to be

$$D_\rho^2\Psi_1 = e^{\frac{1}{2}\rho\mu}\sum_{n=1}^{\infty} A_n(\tfrac{1}{2}\rho)^{\frac{1}{2}}K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho)Q_n(\mu), \quad (3.29)$$

where $K_{n+\frac{1}{2}}(\frac{1}{2}\rho)$ is a modified Bessel function. Since the order of the Bessel function is half-integral, it may be expressed in the closed form

$$(\tfrac{1}{2}\rho)^{\frac{1}{2}}K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}e^{-\frac{1}{2}\rho}\sum_{s=0}^{n}\frac{(n+s)!}{(n-s)!\,\rho^s}. \quad (3.30)$$

The completion of the general integration is rather troublesome (see Goldstein 1929). However, this is not necessary since it is possible to apply further boundary conditions in the partially integrated form (3.29). When (3.29) is expressed in terms of the Stokes variable r, its contribution to $D_r^2\psi$ is seen to be

$$F_1(R)e^{\frac{1}{2}Rr\mu}\sum_{n=1}^{\infty}A_n(\tfrac{1}{2}Rr)^{\frac{1}{2}}K_{n+\frac{1}{2}}(\tfrac{1}{2}Rr)Q_n(\mu). \tag{3.31}$$

Then, since (3.30) shows that

$$(\tfrac{1}{2}\rho)^{\frac{1}{2}}K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho) \sim (\tfrac{1}{2}\pi)^{\frac{1}{2}}(2n)!/\rho^n \tag{3.32}$$

for small values of ρ, the condition that (3.31) should not contain terms of greater order than unity yields the results

$$F_1(R) = R, \tag{3.33}$$

$$A_n = 0 \qquad (n \geqslant 2). \tag{3.34}$$

The contribution to $D_r^2\psi$ is thus

$$-A_1(\tfrac{1}{2}\pi)^{\frac{1}{2}}r^{-1}(1-\mu^2). \tag{3.35}$$

From Stokes's solution (3.15), on the other hand, we obtain

$$D_r^2\psi = \frac{3}{2r}(1-\mu^2)+o(1), \tag{3.36}$$

so that

$$A_1(\tfrac{1}{2}\pi)^{\frac{1}{2}} = -\tfrac{3}{2}. \tag{3.37}$$

The equation (3.29) for Ψ_1 therefore reduces to

$$D_\rho^2\Psi_1 = \frac{3}{4}\left(1+\frac{2}{\rho}\right)e^{-\frac{1}{2}\rho(1-\mu)}(1-\mu^2), \tag{3.38}$$

and the particular integral that is not of greater order than unity in the Stokes region, and whose terms of this order match Stokes's solution, is

$$\Psi_1 = -\tfrac{3}{2}(1+\mu)(1-e^{-\frac{1}{2}\rho(1-\mu)}). \tag{3.39}$$

As was to be expected, this is the rotational part of Oseen's solution (2.5).

3.4. *The second term of the Stokes expansion*

When the first approximation to the left-hand side of the Stokes form (3.1) of the governing equation is calculated from the leading (Stokes) term (3.15), the equation for ψ_1 may be written in the form

$$\frac{f_1(R)}{R}D_r^4\psi_1 = \frac{9}{2}\left(\frac{2}{r^2}-\frac{3}{r^3}+\frac{1}{r^5}\right)Q_2(\mu). \tag{3.40}$$

Hence we may write, for the time being,

$$f_1(R) = R, \tag{3.41}$$

provided we allow for the possibility that the arbitrary constants in the integration of (3.40) may be functions of R. Such functions must, of course, be of smaller order than $1/R$ in order that the solution should not contribute to the leading term of the Stokes expansion; and there is no point in considering functions of smaller order than unity because these will be considered in subsequent terms of the Stokes expansion.

A particular integral of (3.40) which satisfies the boundary conditions at $r = 1$ and $\mu = \pm 1$ is

$$\frac{3}{16}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)Q_2(\mu). \tag{3.42}$$

The general solution satisfying these boundary conditions is then obtained by adding to (3.42) the complementary function (3.20). The contribution that this general solution makes to Ψ is therefore

$$\frac{3R}{16}\left(2\rho^2 - 3R\rho + R^2 - \frac{R^3}{\rho} + \frac{R^4}{\rho_2}\right)Q_2(\mu) +$$
$$+ \sum_{n=1}^{\infty} [A_n\{(2n-1)R^{-n}\rho^{n+3} - (2n+1)R^{-n+2}\rho^{n+1} + 2R^{n+1}\rho^{-n+2}\} +$$
$$+ B_n\{2R^{-n+2}\rho^{n+1} - (2n+1)R^{n+1}\rho^{-n+2} + (2n-1)R^{n+3}\rho^{-n}\}]Q_n(\mu). \tag{3.43}$$

Now, the leading term of the Oseen expansion (3.13) has already been matched to the Stokes expansion, and it has been shown that $F_1(R) \equiv R$. Hence no term in (3.43) must be of greater order than R. Thus we get

$$\left.\begin{aligned} A_n &= 0, \\ B_n &= 0 \qquad (n \geqslant 2), \\ B_1 &= O(1). \end{aligned}\right\} \tag{3.44}$$

When $\rho = O(1)$, (3.43) therefore reduces to

$$R\{\tfrac{3}{8}\rho^2 Q_2(\mu) + 2B_1\rho^2 Q_1(\mu)\} + o(R), \tag{3.45}$$

and B_1 must be chosen so that (3.45) agrees with the expansion of the Oseen term Ψ_1 at small values of ρ. From (3.39), this latter expansion is

$$\Psi_1 = -\tfrac{3}{4}(1-\mu^2)[\rho - \tfrac{1}{4}\rho^2(1-\mu) + O(\rho^3)], \tag{3.46}$$

so that
$$\tfrac{3}{8}Q_2(\mu) + 2B_1 Q_1(\mu) = \tfrac{3}{16}(1-\mu^2)(1-\mu). \tag{3.47}$$

Thus, since
$$Q_2(\mu) = -\tfrac{1}{2}\mu(1-\mu^2), \tag{3.48}$$

we obtain
$$B_1 = -\tfrac{3}{16}. \tag{3.49}$$

The solution for the second term of the Stokes expansion is therefore

$$\psi_1 = \frac{3}{32}\left(2r^2 - 3r + \frac{1}{r}\right)(1-\mu^2) - \frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)(1-\mu^2)\mu. \tag{3.50}$$

It does not seem to have been emphasized sufficiently in the literature that a knowledge of Oseen's solution (2.5) enables a second approximation in the Stokes region to be obtained without a complicated application of the whole Oseen technique for obtaining a second *uniform* approximation. Yet the derivation of this second approximation to ψ is, in principle, a necessary preliminary for the calculation of a second approximation to any property of the flow. The case of the drag coefficient, however, is rather special. For the second term of (3.50), being odd in μ, makes no contribution to the pressure-drag or skin-friction on the sphere. But this term is the particular integral arising from the inertia forces. Hence Oseen's theory, which neglects inertia forces in the Stokes region, must give a proper second approximation to the drag coefficient. The actual calculation of this second approximation is particularly simple because the first term of (3.50) is a multiple ($\frac{3}{8}$) of Stokes's solution. In this way, we obtain Oseen's result (2.6).

Another interesting property which may be discussed on the basis of the second approximation (3.50) is the formation of 'eddies' behind the sphere. The first two terms of the Stokes expansion may be written in the form

$$\psi = \tfrac{1}{4}(r-1)^2(1-\mu^2)\left[\left(1+\frac{3R}{8}\right)\left(2+\frac{1}{r}\right)-\frac{3R}{8}\left(2+\frac{1}{r}+\frac{1}{r^2}\right)\mu\right], \quad (3.51)$$

which, for sufficiently small values of R, vanishes only at $r = 1$ and $\mu = \pm 1$. For larger values of R, however, ψ also vanishes along the (real) curve whose polar equation is (according to (3.51))

$$\mu = \left(\frac{8}{3R}+1\right)\frac{2r^2+r}{2r^2+r+1}, \quad (3.52)$$

and this curve is the boundary of the 'eddies'. Equation (3.52) shows that the minimum value of μ occurs at $r = 1$ and is

$$\mu_{\min} = \frac{2}{R}+\frac{3}{4}, \quad (3.53)$$

so that the 'eddies' first appear at the rear stagnation point and do so when $R = 8$. Of course, this Reynolds number is far too large to make estimates based on only two terms of the Stokes expansion at all reliable. In fact, it cannot seriously be claimed that slow-motion theory gives even a qualitative explanation of the phenomenon. A similar conclusion was reached by Pearcey & McHugh (1955) in the case of Oseen's (rather than the Navier–Stokes) equation, after a careful numerical evaluation of Goldstein's (1929) exact solution*.

3.5. *Higher approximations*

Since higher-order terms in the Stokes and Oseen expansions are not proportional to simple powers of R, it seems desirable to give a brief account of the nature of these terms. This can be done most easily by first considering the third term of the Stokes expansion, ψ_2.

* In this respect, the numerical work of Tomotika & Aoi (1950) appears to be seriously in error.

If we set
$$f_2(R) = R^2 \tag{3.54}$$
and allow the arbitrary constants in the integration for ψ_2 to be functions of the Reynolds number, in the manner described for the case of ψ_1, an elementary application of the matching conditions shows that the general solution for ψ_2 must be of the form

$$\begin{aligned}\psi_2 = &-\frac{3}{40}\left(C_1 r^2 + C_2 r + \frac{C_3}{r} - r^3 + 3r^2\log r - \frac{3}{4} - \right.\\ &\left. -\frac{3\log r}{5r} - \frac{7}{24r^2} + \frac{1}{40r^3}\right)Q_1(\mu) + \\ &+\frac{27}{32}\left(C_4 r^3 + C_5 + \frac{C_6}{r^2} + \frac{r^2}{3} - \frac{r}{2} - \frac{1}{6r}\right)Q_2(\mu) + \\ &+\frac{9}{20}\left(\frac{C_7}{r} + \frac{C_8}{r^3} + \frac{r^3}{9} - \frac{43r^2}{120} + \frac{11r}{24} - \frac{1}{3} + \frac{4\log r}{35r} + \frac{\log r}{42r^3}\right)Q_3(\mu), \end{aligned}\tag{3.55}$$

where C_n are constants of integration. The evaluation of these constants proceeds as follows. Both C_7 and C_8 follow immediately from the no-slip condition on the sphere. The term in C_4 makes a contribution $\frac{27}{32}C_4 R\rho^3$ to Ψ, so that C_4 may be evaluated from the small-ρ expansion of the (known) function Ψ_1; C_5 and C_6 then follow from the no-slip condition. The case of C_1, C_2 and C_3, however, is somewhat different owing to the presence of the particular integral in $r^2\log r$. In fact, the contribution to Ψ of the term in $Q_1(\mu)$ is
$$-\tfrac{3}{40}(-R\rho^3 - 3R^2\log R\rho^2 + 3R^2\rho^2\log\rho + C_1 R^2\rho^2)Q_1(\mu) + o(R^2), \tag{3.56}$$
which contains a term in $R^2\log R$. If, therefore, there is no term of this form in the Oseen expansion (and it will be shown below that this is indeed the case), we must have
$$C_1 = 3\log R + O(1). \tag{3.57}$$
The constants C_2 and C_3 must then be multiples of $\log R$ also, in order to satisfy the no-slip condition on the sphere. Indeed, it is clear that we should replace (3.54) by
$$f_2(R) = R^2\log R \tag{3.58}$$
and that the corresponding ψ_2 is a finite multiple of Stokes's solution (3.15). Thus, using (3.55) and (3.57),
$$\psi_2 = \frac{9}{160}\left(2r^2 - 3r + \frac{1}{r}\right)(1-\mu^2). \tag{3.59}$$

The proof that there is no term in $R^2\log R$ in the Oseen expansion is very simple. Such a term would have to satisfy Oseen's equation (3.26) and the general relevant solution for $D_\rho^2\Psi_2$ would be (3.29), in which the coefficients A_n are of order unity. But the contributions to $D_r^2\,\psi$ would then be
$$(R^2\log R)e^{\frac{1}{2}Rr\mu}\sum_{n=1}^{\infty} A_n(\tfrac{1}{2}Rr)^{\frac{1}{2}}K_{n+\frac{1}{2}}(\tfrac{1}{2}Rr)Q_n(\mu) \tag{3.60}$$
and (3.32) shows that the order of magnitude of (3.60) in the Stokes region is $R\log R$ or greater. Such terms are known not to exist, so all the A_n

must be zero. A similar proof applies to the solution of the resulting equation

$$D_\rho^2 \Psi_2 = 0.$$

Hence there is no term in $R^2 \log R$ (or, indeed, in any function of R whose order of magnitude lies between R and R^2) in the Oseen expansion, and (3.58) and (3.59) are a valid representation of the third term of the Stokes expansion.

Nevertheless, the Oseen expansion must ultimately involve terms in $\log R$. For the strength of the source-flow outside the wake in the Oseen region is known to be proportional to the drag coefficient (Goldstein 1929). And, according to the first three terms of the Stokes expansion, the drag coefficient is

$$C_D = \frac{6\pi}{R}\left(1 + \frac{3}{8}R + \frac{9}{40}R^2 \log R + O(R^2)\right) \tag{3.61}$$

(the third term is a new result). The argument shows, in fact, that the first occurrence of $\log R$ in the Oseen expansion is in the term $R^3 \log R$. The non-linearity of the Navier–Stokes equation then shows that both expansions must involve powers of $\log R$, and it seems reasonable to suppose that both expansions are in powers of R, each term of which is multiplied by a polynomial in $\log R$.

Finally, it is worth noting that, although the technique was not designed with such in mind, the Oseen expansion is actually uniformly valid over the whole field of flow. Each successive term of the expansion does not give a higher uniform approximation, but it is always possible to find a *finite* number of terms which give, to any required accuracy, a uniform approximation to the solution. For instance, a first approximation requires all the terms up to, and including those in R^3, since, at that stage, the last term (the doublet) of Stokes's solution has been matched. The fact that it is possible to choose a *finite* number of terms in this way provides a very simple *à posteriori* justification for the formal matching procedure.

4. General expansions for flow past a circular cylinder

In this section, we attempt to obtain higher approximations to the velocity distribution for flow past a circular cylinder, and to discover how far the method adopted for the sphere in § 3 will apply in this case. This problem has been considered in outline by Lagerstrom & Cole (1955) who introduce Stokes and Oseen variables and obtain Stokes and Oseen expansions that follow naturally from the limit processes they adopt. This present account considers the problem in rather more detail, in order to display more fully the structure of the expansions*. In § 4.5 some comments

* At a late stage in the preparation of this paper, a short paper by Kaplun was read at the IX International Congress of Applied Mechanics entitled " Low Reynolds Number Flow Past a Circular Cylinder ". Though by no means a detailed account, it appears to present the same conclusions as are reached in the present paper, and gives an improved approximation for the drag.

are made on the effect of those inertia terms that are transcendentally small compared with the expansion parameter.

4.1. *The Stokes and Oseen expansions*

In the Stokes region of the flow, the equation for the stream function becomes

$$\nabla_r^4 \psi = -\frac{R}{r}\frac{\partial(\psi, \nabla_r^2 \psi)}{\partial(r, \theta)} \tag{4.1}$$

where r, θ, and ψ are as defined in § 2.2, and where

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{4.2}$$

We assume an expansion of the form

$$\psi = f_0(R)\psi_0(r, \theta) + f_1(R)\psi_1(r, \theta) + f_2(R)\psi_2(r, \theta) + \dots, \tag{4.3}$$

where
$$f_{n+1}(R)/f_n(R) \to 0 \quad \text{as} \quad R \to 0 \tag{4.4}$$

and $f_0(R)$ remains bounded at $R = 0$. As in the three-dimensional case, the expansion (4.3) is to satisfy the equation (4.1) and the no-slip condition on the cylinder, while the condition at infinity must be replaced by the condition that the expansion should match an expansion which is valid in the outer region. Again, for similar reasons, we introduce Oseen variables

$$\rho = Rr, \qquad \Psi = R\psi, \tag{4.5}$$

in terms of which the equation (4.1) becomes

$$\nabla_\rho^4 \Psi = -\frac{1}{\rho}\frac{\partial(\Psi, \nabla_\rho^2 \Psi)}{\partial(\rho, \theta)}. \tag{4.6}$$

We now assume an expansion

$$\Psi = \Psi_0(\rho, \theta) + F_1(R)\Psi_1(\rho, \theta) + F_2(R)\Psi_2(\rho, \theta) + \dots, \tag{4.7}$$

where
$$F_{n+1}(R)/F_n(R) \to 0 \quad \text{as} \quad R \to 0. \tag{4.8}$$

The expansion (4.7) is to satisfy the equation (4.6), the uniform stream condition at infinity, and, as before, a matching requirement for small values of ρ: that it should match the Stokes expansion (4.3).

4.2. *The leading terms of the expansions*

From the approach described in § 2.2, it follows that we may expect the leading term ψ_0 of the expansion (4.3) to be the Stokes solution (2.7) with $f_0(R) = C$, and the leading term Ψ_0 of the expansion (4.7) to be the uniform stream

$$\Psi_0 = \rho \sin \theta. \tag{4.9}$$

That (4.9) is in fact correct can be deduced in exactly the same way that Ψ_0 was obtained for the sphere in § 3.2. Similarly, we can say that ψ_0 must satisfy

$$\nabla_r^4 \psi_0 = 0 \tag{4.10}$$

since $f_0(0)$ is bounded. The most general solution of (4.10) that is antisymmetric about $\theta = 0$ is

$$\psi_0 = [B_1(2r\log r - r + r^{-1}) + C_1(r^3 - 2r + r^{-1})]\sin\theta + \\ + \sum_{n=2}^{\infty} [B_n(r^{n+2} - (n+1)r^{2-n} + nr^{-n}) + C_n(r^n - nr^{2-n} + (n-1)r^{-n})]\sin n\theta, \tag{4.11}$$

where B_n, C_n $(n \geqslant 1)$ are constants. When (4.11) is expressed in terms of the Oseen variable ρ the contribution of $f_0(R)\psi_0$ to Ψ becomes

$$f_0(R)\Big\{[B_1(2\rho\log\rho - 2\rho\log R - \rho + R^2\rho^{-1}) + C_1(\rho^3R^{-2} + R^2\rho^{-1} - 2\rho)]\sin\theta + \\ + \sum_{n=2}^{\infty} [B_n(\rho^{n+2}R^{-n-1} - (n+1)\rho^{2-n}R^{n-1} + n\rho^{-n}R^{n+1}) + \\ + C_n(\rho^nR^{-n+1} - n\rho^{2-n}R^{n-1} + (n-1)\rho^{-n}R^{n+1})]\sin n\theta\Big\}. \tag{4.12}$$

If this is not to contain any terms of greater order than unity, then

$$\left.\begin{aligned} C_1 &= 0, \\ B_n &= 0, \\ C_n &= 0 \qquad (n \geqslant 2), \\ f_0(R) &= 1/\log R. \end{aligned}\right\} \tag{4.13}$$

Hence (4.12) becomes

$$-2B_1\rho\sin\theta + O(1/\log R), \tag{4.14}$$

and because of the matching condition and (4.9) we get

$$B_1 = -\tfrac{1}{2}. \tag{4.15}$$

In view of the non-linearity of the Navier–Stokes equations, the term of order $(\log R)^{-1}$ in (4.14) suggests that the functions $F_n(R)$ will be inverse powers of $\log R$.

4.3. *The second term of the Oseen expansion*

Having established that Ψ_0 represents a uniform stream, we can use Oseen's equation to solve for Ψ_1. In terms of the stream function this is

$$\left(\nabla_\rho^2 - \frac{\partial}{\partial\xi}\right)\nabla_\rho^2\Psi_1 = 0, \tag{4.16}$$

where

$$\xi = \rho\cos\theta. \tag{4.17}$$

A first integral to this equation that is bounded at infinity is given by

$$\nabla_\rho^2\Psi_1 = e^{\frac{1}{2}\xi}\sum_{n=1}^{\infty} X_n K_n(\tfrac{1}{2}\rho)\sin n\theta \tag{4.18}$$

where X_n are constants and $K_n(\frac{1}{2}\rho)$ is a modified Bessel function. Without solving for Ψ_1 we may apply the matching requirement directly to the vorticity, $\nabla_\rho^2\Psi_1$, in the form (4.18). We know, from the Stokes solution ψ_0, that

$$f_0(R)\nabla_r^2\psi_0 = \frac{2}{r\log R}\sin\theta. \tag{4.19}$$

Writing (4.18) in terms of the Stokes variable r, we get

$$\nabla_\rho^2 \Psi_1 = e^{\frac{1}{2}Rr\cos\theta} \sum_{n=1}^{\infty} X_n K_n(\tfrac{1}{2}Rr)\sin n\theta. \tag{4.20}$$

Since the $K_n(z)$ behave as z^{-n} for small values of z, and since $R\nabla_\rho^2\Psi$ must match $\nabla_r^2\psi$, we must put

$$X_1 = 1, \qquad X_n = 0 \quad \text{for} \quad n > 1, \tag{4.21}$$

and

$$F_1(R) = (\log R)^{-1}. \tag{4.22}$$

Equation (4.18) now becomes

$$\nabla_\rho^2 \Psi_1 = e^{\frac{1}{2}\rho\cos\theta} K_1(\tfrac{1}{2}\rho)\sin\theta \tag{4.23}$$

and this may be integrated (see Tomotika & Aoi 1950) to give

$$\Psi_1 = \sum_{n=1}^{\infty} \phi_n(\tfrac{1}{2}\rho)\frac{\rho \sin n\theta}{n} + \text{harmonic function}, \tag{4.24}$$

where

$$\phi_n = 2K_1 I_n + K_0(I_{n+1} + I_{n-1}), \tag{4.25}$$

the K_m and I_m being modified Bessel functions. Now Ψ_1 must tend to zero for large values of ρ and so the harmonic function in (4.24) can only be of the form

$$Y_n \rho^{-n} \sin n\theta.$$

The matching condition between $R\psi$ and Ψ then requires that the constants Y_n must vanish for all n. Thus (4.24) reduces to the relevant part of Lamb's result (2.13).

4.4. *Higher terms in the Stokes and Oseen expansions*

From what has been shown above, it is clear that

$$F_n(R) = (\log R)^{-n}. \tag{4.26}$$

It is not difficult to see that the Stokes solution given by (4.11) and (4.13) will not include terms of order $(\log R)^{-2}$, $(\log R)^{-3}$, etc. Hence we conclude that

$$f_n(R) = (\log R)^{-n-1} \tag{4.27}$$

with the consequent result that

$$\nabla_r^4 \psi_n = 0 \quad \text{for all } n.$$

This means that as far as the Stokes expansion is concerned, the inertial terms are never important enough to be considered in the governing equations; their effect is felt only through the outer boundary or matching condition. All the arguments applied to the general solution for ψ_0 will now apply to the general solutions for ψ_n, and we can infer that the ψ_n differ from one another only by a numerical factor. This means that the Stokes expansion has now been reduced to the form,

$$\psi = f(R)\psi_0 + O(R), \tag{4.28}$$

where

$$\psi_0(r,\theta) = (2r\log r - r + r^{-1})\sin\theta \tag{4.29}$$

and

$$f(R) = \alpha_1(\log R)^{-1} + \alpha_2(\log R)^{-2} + \ldots, \tag{4.30}$$

the α_n being constants. So far we have determined $\alpha_1 = -\frac{1}{2}$. The next step is therefore to solve for Ψ_2. This is given by the equation

$$\left(\nabla_\rho^2 - \frac{\partial}{\partial \xi}\right)\nabla_\rho^2 \Psi_2 = -\frac{1}{\rho}\frac{\partial(\Psi_1, \nabla^2\Psi_1)}{\partial(\rho,\theta)} \tag{4.31}$$

and must vanish at infinity. The calculation is straightforward, the solution involving a particular integral from the right-hand side of (4.31) and a complementary function given by (4.18). If $\nabla_\rho^2\Psi_1$ is expressed in the form (4.23) and Ψ_1 in the form (4.24) then we obtain readily

$$-\frac{1}{\rho}\frac{\partial(\Psi_1, \nabla_\rho^2\Psi_1)}{\partial(\rho,\theta)} = e^{\frac{1}{2}\xi}\sum_{n=1}^{\infty} g_n(\rho)\sin n\theta \tag{4.32}$$

where

$$\left.\begin{aligned} g_1(R) &= O(R^{-1}),\\ g_2(R) &= O(R^{-2}),\\ g_p(R) &= O(R^{p-2}) \qquad (p > 2). \end{aligned}\right\} \tag{4.33}$$

If we write $\nabla_\rho^2\Psi_2 = e^{\frac{1}{2}\xi}\Pi$, then (4.31) and (4.32) give

$$(\nabla_\rho^2 + \tfrac{1}{4})\Pi = \sum_{n=1}^{\infty} g_n(\rho)\sin n\theta. \tag{4.34}$$

Solving (4.34) by the method of variation of parameters we get

$$\nabla_\rho^2\Psi_2 = \sum_{n=1}^{\infty} k_n(\rho)\sin n\theta + \text{complementary function}, \tag{4.35}$$

where $k_n(R)$ is $O(R^{n-2})$ except for $n = 1$ when it is $O(R)$. By the same argument as that used in §4.3, the complementary function in (4.35) must be $Z_1 e^{\frac{1}{2}\rho\cos\theta} K_1(\frac{1}{2}\rho)\sin\theta$, and hence (4.35) becomes

$$\nabla_\rho^2\Psi_2 = \sum_{n=1}^{\infty} k_n(\rho)\sin n\theta + Z_1 e^{\frac{1}{2}\rho\cos\theta} K_1(\tfrac{1}{2}\rho)\sin\theta. \tag{4.36}$$

Finally, on integrating again, we get

$$\Psi_2 = \sum_{n=1}^{\infty} Z_1\phi_n(\tfrac{1}{2}\rho)\frac{\rho\sin n\theta}{n} + \sum_{n=1}^{\infty} l_n(\rho)\sin n\theta \tag{4.37}$$

where

$$l_n(R) = O(R^n). \tag{4.38}$$

and in particular $l_1(R) = L_1(R) + O(R^2)$ where L_1 is a constant. When $F_2(R)\Psi_2$ is written in terms of the Stokes coordinate r, it is found to give a contribution $-Z_1(\log R)^{-1}r$ to $f_0(R)\psi_0$. Thus, from the matching requirement, we deduce that

$$Z_1 = (\tfrac{1}{2} - \gamma + \log 4). \tag{4.39}$$

Next, the vorticity term $\nabla_\rho^2\Psi_2$ given by (4.36) may be matched to the corresponding vorticity term $R^{-1}\nabla_r^2\psi_1$ in the way that (4.19) was matched to (4.20), to give

$$\alpha_2 = -\tfrac{1}{2}(\tfrac{1}{2} - \gamma + \log 4), \tag{4.40}$$

α_2 being the coefficient of $(\log R)^{-2}$ in (4.30).

It is not difficult to see how the process outlined above to give Ψ_2 and thence α_2 may be used successively to obtain all the Ψ_p, α_p. Each of the Ψ_p will consist of a complementary function

$$R_p \sum_{n=1}^{\infty} \phi_n(\tfrac{1}{2}\rho)\frac{\rho \sin n\theta}{n},$$

where R_p is some constant, and a particular integral

$$\sum_{n=1}^{\infty} m_{pn}(\rho)\sin n\theta,$$

where the $m_{pn}(\rho)$ display the same behaviour as the $l_n(\rho)$ in (4.37).

4.5. *Further effects of the intertia terms*

The expansions (4.3) and (4.7) which have been considered above differ very greatly from the corresponding expansions (3.5) and (3.13) obtained for the sphere. In fact, when (4.3) is written in the form (4.28) it is seen that the entire Stokes expansion obtained for the cylinder corresponds to just the first term of the Stokes expansion for the sphere. Close to the cylinder, virtually no account has been taken of the inertia terms, while in the Oseen region terms of order R have been neglected.

From a physical point of view the expansions that we have obtained cannot be expected to provide much information. Many important characteristics of slow motion will be caused by just those terms of order R that have been neglected because R is transcendentally small compared with any power of $(\log R)^{-1}$. We can attempt, in a formal way, to take account of these terms of order R (and also of higher powers of R) by writing our expansions for the stream function in the form

$$\begin{aligned}\psi = {} & (\log R)^{\beta_0}[\psi_{00}(r,\theta)+(\log R)^{-1}\psi_{01}(r,\theta)+(\log R)^{-2}\psi_{02}(r,\theta)+\dots]+\\ & + R(\log R)^{\beta_1}[\psi_{10}(r,\theta)+(\log R)^{-1}\psi_{11}(r,\theta)+(\log R)^{-2}\psi_{12}(r,\theta)+\dots]+\\ & \quad \cdots\cdots\cdots\\ & + R^n(\log R)^{\beta_n}[\psi_{n0}(r,\theta)+(\log R)^{-1}\psi_{n1}(r,\theta)+(\log R)^{-2}\psi_{n2}(r,\theta)+\dots]\\ & \quad \cdots\cdots\cdots \end{aligned} \tag{4.41}$$

for the Stokes expansion, where $\beta_0, \beta_1, \dots \beta_n$ are integers, with a similar form for the Oseen expansions.

These expansions would have to satisfy the same equations, the same boundary conditions and the same matching condition as in § 4.1. Formally, the derivation of the terms ψ_{0p} and Ψ_{0p} would be carried out exactly as in § 4.2, § 4.3 and § 4.4. It is seen readily that the ψ_{0p} substituted into the right-hand side of (4.1) would yield inhomogeneous equations for the ψ_{1p}. The matching conditions with the boundary conditions would then enable the arbitrary constants in the integration to be determined. By a process very much like that used for the ψ_{0p} and Ψ_{0p}, all the ψ_{jk} and Ψ_{ln} could be derived successively. It will be observed that only a finite number of the

ψ_{jk} and Ψ_{ln} would be needed in order to obtain any particular one of them; in this sense, the formal process suggested above for determining the various functions in (4.41) is a possible one.

Asymptotic expansions for low Reynolds number flow are used extensively to give numerical approximations to certain constants of the physical flow. For R as small as 10^{-2}, R is larger than $(\log R)^{-3}$, and so from a numerical point of view the term ψ_{10} in (4.41) might well be more important than ψ_{03}. Though it is difficult to justify in any analytic sense the use of (4.41), it is nevertheless clear that many properties of slow streaming motion past a circular cylinder which do not form any part of the asymptotic expansions (4.3) and (4.7) might be predicted by using a suitable number of terms from the conjectural expansion (4.41).

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